ISRAEL JOURNAL OF MATHEMATICS, Vol. 70, No. 1, 1990

AN EXAMPLE OF A NON-LERF GROUP WHICH IS A FREE PRODUCT OF LERF GROUPS WITH AN AMALGAMATED CYCLIC SUBGROUP

BY
E. RIPS
Institute of Mathematics and Computer Sciences,
The Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel

To My Mother

ABSTRACT
A group with the properties of the title is constructed.

For a group G and an element $g \in G$ we denote

$$I(G, g) = \{n \in \mathbf{N} \mid \exists R \trianglelefteq_f G \text{ s.t. } R \cap \langle g \rangle = \langle g^n \rangle\}.$$

For two groups A, B and elements $a \in A$, $b \in B$ of the same order we introduce the notation

$$(G, g) = (A, a) * (B, b)$$

which means that $G = A \underset{a=b}{*} B$ and $g = i_1(a) = i_2(b)$ where $i_1 : A \hookrightarrow G$, $i_2 : B \hookrightarrow G$ are the standard inclusions. If $(G_i)_{i \in S}$ is a family of groups and $g_i \in G_i$ are elements of the same order for all $i \in S$ then

$$(G, g) = \underset{i \in S}{*} (G_i, g_i)$$

has a similar meaning.

$\leqq_f$ reads "subgroup of finite index" and $\trianglelefteq_f$ reads "normal subgroup of finite index".

A group G is called LERF (locally extended residually finite) if for any

Received June 4, 1989

elements $r_0, r_1, \ldots, r_k \in G$ such that $r_0 \notin \langle r_1, \ldots, r_k \rangle$ there is a subgroup of finite index in G containing $r_1, \ldots, r_k$ but not r_0.

We deduce the existence of a group with the properties described in the title from the following two propositions.

PROPOSITION 1. *Let $(G_i)_{i \in S}$ be a family of groups and let $g_i \in G_i$ be elements of an infinite order. For any subset $T \subseteq S$ let*

$$(G_T, g_T) = \mathop{*}_{i \in T} (G_i, g_i).$$

Suppose that the following conditions hold:

(1) *for every finite subset $T \subseteq S$, the group G_T is LERF;*
(2) *for each $n \in \mathbf{N}$ there exists $m \in \bigcap_{i \in S} I(G_i, g_i)$ such that $n \mid m$;*
(3) *for each $m \in \bigcap_{i \in S} I(G_i, g_i)$ there exists a family of groups $(N_i)_{i \in S}$ such that*
   (i) $N_i \trianglelefteq_f G_i$;
   (ii) $N_i \cap \langle g_i \rangle = \langle g_i^m \rangle$;
   (iii) *there are only finitely many isomorphism types of pairs $(G_i/N_i, g_iN_i)$ as i runs over S.*

Then G_S is LERF.

PROPOSITION 2. *Let $m_0 = 1 < m_1 < m_2 < m_3 < \cdots$ be a sequence of natural numbers such that $m_{i-1} \mid m_i$ for all $i \in \mathbf{N}$. Then there exists a family of groups $(G_i)_{i \in \mathbf{N}}$ and elements $g_i \in G_i$ of infinite order with the following properties:*

(1) *each G_i is a finitely generated group with an abelian subgroup of finite index;*
(2) $I(G_i, g_i) = \{m_0 = 1, m_1, m_2, \ldots, m_{i-1}, m_i, 2m_i, 3m_i, \ldots\}$;
(3) *for $k = 0, 1, 2, \ldots, i-1$, there is a unique normal subgroup N_{ik} of finite index in G_i such that*

$$N_{ik} \cap \langle g_i \rangle = \langle g_i^{m_k} \rangle;$$

(4) *for every $j > i$, $G_i/N_{ik} \cong G_j/N_{jk}$, and the isomorphism takes g_iN_{ik} to g_jN_{jk}.*

Given these two propositions, we proceed as follows. Choose two sequences (m_i) and (n_i) such that

(1) $m_0 = 1 < m_1 < m_2 < \cdots$, $n_0 = 1 < n_1 < n_2 < \cdots$;
(2) for all $i \in \mathbf{N}$, $m_{i-1} \mid m_i$ and $n_{i-1} \mid n_i$;
(3) for all $n \in \mathbf{N}$ there is $i \in \mathbf{N}$ such that $n \mid m_i$ and $n \mid n_i$;
(4) $m_i \neq n_j$ whenever $i > 0$ or $j > 0$.

Let $(G_i, g_i)_{i\in\mathbf{N}}$ and $(H_i, h_i)_{i\in\mathbf{N}}$ be families satisfying conditions (1), (2), (3), (4) of Proposition 2 for (m_i) and (n_i) respectively.

Every finitely generated group with an abelian normal subgroup of finite index is LERF, so by (1) each G_i and H_i is LERF. If there is some finite set $T \subseteq \mathbf{N}$ such that G_T is not LERF, then taking a minimal such T and considering a partition $T = T' \cup T''$, with T' and T'' non-empty we obtain

$$(G_T, g_T) \cong (G_{T'}, g_{T'}) * (G_{T''}, g_{T''})$$

where $G_{T'}$ and $G_{T''}$ are LERF, which yields the desired example.

So let us assume that for any finite subset $T \subseteq \mathbf{N}$, G_T and H_T are LERF. In view of condition (2),

$$\bigcap_{i\in\mathbf{N}} I(G_i, g_i) = \{m_0, m_1, \ldots, m_k, \ldots\}, \qquad \bigcap_{i\in\mathbf{N}} I(H_i, h_i) = \{n_0, n_1, \ldots, n_k, \ldots\}.$$

Then by (3), condition (2) of Proposition 1 holds both for (m_i) and (n_i). It is also clear that conditions (3) and (4) of Proposition 2 imply condition (3) of Proposition 1. Then, by Proposition 1, $G_{\mathbf{N}}$ and $H_{\mathbf{N}}$ are LERF. But for

$$(A, a) = (G_{\mathbf{N}}, g_{\mathbf{N}}) * (H_{\mathbf{N}}, h_{\mathbf{N}})$$

we have by (4)

$$\begin{aligned} I(A, a) &\subseteq I(G_{\mathbf{N}}, g_{\mathbf{N}}) \cap I(H_{\mathbf{N}}, h_{\mathbf{N}}) \\ &\subseteq \bigcap_{i\in\mathbf{N}} I(G_i, g_i) \cap \bigcap_{i\in\mathbf{N}} I(H_i, h_i) = \{1\}, \end{aligned}$$

which means that A is not residually finite, hence not LERF.

Proof of Proposition 1. Whenever convenient, for $T_1 \subseteq T_2$ we identify G_{T_1} with a subgroup of G_{T_2}.

Let $r_0, r_1, \ldots, r_k \in G_S$, $r_0 \notin \langle r_1, \ldots, r_k \rangle$. Then for some finite subset $T \subseteq S$, we have $r_0, r_1, \ldots, r_k \in G_T$. By condition (1), G_T is LERF, so there exists $R \leqq_f G_T$ such that $r_0 \notin R \supseteq \langle r_1, \ldots, r_k \rangle$. Let

$$C = \bigcap_{x\in G_T} x^{-1}Rx.$$

Then $C \leqq_f G_T$ and for some n, $C \cap \langle g_T \rangle = \langle g_T^n \rangle$. By condition (2), one can find $m \in \bigcap_{i\in S} I(G_i, g_i)$ such that $n \mid m$. According to condition (3), there is a family $(N_i)_{i\in S}$ satisfying (i), (ii), (iii). Without loss of generality we can assume that

(iv) $N_i \subseteq C$ for each $i \in T$.

For any subset $L \subseteq S$, let N_L denote the normal subgroup of G_L generated by all $N_j, j \in L$. Then, obviously,

$$(5) \qquad (G_S/N_S, g_S N_S) \cong (G_T/N_T, g_T N_T) * (G_{S\setminus T}/N_{S\setminus T}, g_{S\setminus T}N_{S\setminus T}).$$

By (iv), $N_T \subseteq C \subseteq R$ and therefore in G_T/N_T

$$(6) \qquad r_0 N_T \notin \langle r_1 N_T, \ldots, r_k N_T \rangle.$$

According to (iii), there are finitely many pairs

$$(A_1, a_1), (A_2, a_2), \ldots, (A_l, a_l)$$

such that each $(G_i N_i, g_i N_i)$ is isomorphic to one of them. Let

$$(A, a) = \mathop{*}_{h=1}^{l} (A_i, a_i).$$

We have then a homomorphism

$$(7) \qquad \lambda : G_{S\setminus T}/N_{S\setminus T} \to A$$

such that $\lambda(g_{S\setminus T}N_{S\setminus T}) = a$. Omitting, if necessary, some factors A_i, we can assume that λ is onto. Let

$$(B, b) = (G_T/N_T, g_T N_T) * (A, a).$$

By (5) and (7), we have an epimorphism

$$\mu : G_S/N_S \to B$$

such that $\mu(g_S N_S) = b$, $\mu|_{G_T/N_T} = \mathrm{Id}$, $\mu|_A = \lambda$.

The group B is an amalgamated free product of finitely many finite groups, hence it is LERF (see [AG], Lemma 3). In view of (6), there exists $Q \leqq_f B$ such that

$$r_0 N_T \notin Q \geqq \langle r_1 N_T, \ldots, r_k N_T \rangle.$$

Let $\varphi : G_S \to B$ be the composition of the natural homomorphism $G_S \to G_S/N_S$ and $\mu : G_S/N_S \to B$. Then $\varphi^{-1}(Q) \leqq_f G_S$, $r_0 \notin Q \supseteq \langle r_1, \ldots, r_k \rangle$. Therefore G_S is LERF. The proposition is proved.

Proof of Proposition 2. For $i \in \mathbf{N}$, let A_i be a finite simple group with some fixed element $a_i \in A_i$ of order m_i/m_{i-1}. We define inductively

$$(8) \qquad B_0 = \{1\}, \quad B_i = A_i \operatorname{wr} B_{i-1} \qquad (i \geqq 1).$$

In the wreath product A wr B the copy of the copy A that corresponds to an element $b \in B$ will be denoted $A(b)$ and its elements will be denoted by $a(b)$, $a \in A$. In this notation, for any $a \in A$, $b_1, b_2 \in B$ we have $b_2^{-1}a(b_1)b_2 = a(b_1 b_2)$. For more information on wreath products see, for example, [Ne], Ch.2.

We define elements $b_i \in B_i$ as follows:

(9) $$b_0 = 1, \quad b_i = a_i(1)b_{i-1} \qquad (i \geqq 1).$$

Let $C = \langle c \rangle$ be an infinite cyclic group. We take

$$C_i = C \operatorname{wr} B_i \qquad (i \geqq 1).$$

Then $C_i = C^{B_i} \rtimes B_i$. It is well known that there is an isomorphism of B_i-modules

(10) $$\mu : C^{B_i} \to \mathbf{Z}[B_i]$$

where B_i acts on C^{B_i} by conjugation and on the additive group of the integral group ring $\mathbf{Z}[B_i]$ by multiplication on the right. We have homomorphism $\lambda : B_i \to B_{i-1}$ with $\operatorname{Ker} \lambda = A_i^{B_{i-1}}$. It induces a homomorphism of group rings

$$\bar{\lambda} : \mathbf{Z}[B_i] \to \mathbf{Z}[B_{i-1}].$$

It is well known that $\operatorname{Ker} \bar{\lambda} = \Delta(\operatorname{Ker} \lambda)\mathbf{Z}[B_i] = \Delta(A_i^{B_{i-1}})\mathbf{Z}[B_i]$, where $\Delta(G)$ denotes the augmentation ideal of the group ring $\mathbf{Z}[G]$.

We take

(11) $$D_i = \mu^{-1}(\operatorname{Ker} \bar{\lambda}), \quad G_i = D_i \rtimes B_i \subseteq C \operatorname{wr} B_i \qquad (i \geqq 1).$$

Let $d \in A_i^{B_{i-1}}$, $d \neq 1$. Then, clearly, $c(d)c(1)^{-1} \in D_i$ and we define

(12) $$g_i = c(d)c(1)^{-1}b_i \in G_i \qquad (i \geqq 1).$$

We have to show that $(G_i, g_i)_{i \in \mathbf{N}}$ satisfy all the conditions of Proposition 2.

By the construction, each G_i is a finitely generated group with an abelian subgroup of finite index, so condition (1) is satisfied.

For the rest, we need a few lemmas.

LEMMA 1. *$[D_i, A_i^{B_{i-1}}] = D_i$ for all $i \geqq 1$.*

PROOF. It is well-known (see, for example, [Gr], §2.4) that for any group G

$$\Delta(G)/\Delta(G)^2 \cong G/G' = G^{ab}.$$

Since A_i is simple, we have $(A_i^{B_{i-1}})^{ab} = \{1\}$, hence

$$\Delta(A_i^{B_{i-1}})^2 = \Delta(A_i^{B_{i-1}}).$$

For any $x \in D_i$, $y \in A_i^{B_{i-1}}$

$$\mu([x, y]) = \mu(x^{-1}y^{-1}xy) = -\mu(x) + \mu(x)y = \mu(x)(-1 + y).$$

We have

$$\begin{aligned}(\Delta(A_i^{B_{i-1}})\mathbf{Z}[B_i])\Delta(A_i^{B_{i-1}})) &= (\mathbf{Z}[B_i]\Delta(A_i^{B_{i-1}}))\Delta(A_i^{B_{i-1}}) \\ &= \mathbf{Z}[B_i]\Delta(A_i^{B_{i-1}})^2 = \mathbf{Z}[B_i]\Delta(A_i^{B_{i-1}}) = \Delta(A_i^{B_{i-1}})\mathbf{Z}[B_i],\end{aligned}$$

hence the elements $\mu(x)(-1 + y)$ generate additively the ideal $\Delta(A_i^{B_{i-1}})\mathbf{Z}[B_i] = \operatorname{Ker}\bar{\lambda}$ and therefore the commutators $[x, y]$ with $x \in D_i$, $y \in A_i^{B_{i-1}}$, generate $D_i = \mu^{-1}(\operatorname{Ker}\bar{\lambda})$. □

For $k = 0, 1, \ldots, i$ consider the (obvious) homomorphisms $\varphi_{ik} : B_i \to B_k$ and $\psi_{ik} : G_i \to B_k$ and let $M_{ik} = \operatorname{Ker}\varphi_{ik}$, $N_{ik} = \operatorname{Ker}\psi_{ik}$.

Lemma 2. *Every normal subgroup of B_i is of the form M_{ik}, $0 \leqq k \leqq i$.*

Proof. For $i > 1$, $B_i = A_i^{B_{i-1}} \rtimes B_{i-1}$. Since A_i is simple every non-trivial normal subgroup of B_i contains $A_i^{B_{i-1}}$, and our claim follows by induction on i. □

Lemma 3. *For any $N \trianglelefteq G_i$, either $N \subseteq D_i$ or $N = N_{ik}$ for some k, $0 \leqq k \leqq i$.*

Proof. In view of Lemma 2, we have only to show that if $N \not\subseteq D_i$ then $D_i \subseteq N$. So assume that $N \not\subseteq D_i$. By Lemma 2, $\psi_{ii}(N)$ contains $M_{i\,i-1} = A_i^{B_{i-1}}$. By Lemma 1, $[D_i, A_i^{B_{i-1}}] = D_i$. Since D_i is abelian and $D_iN \supseteq A_i^{B_{i-1}}$, we have

$$[D_i, N] = [D_i, D_iN] \supseteq [D_i, A_i^{B_{i-1}}] = D_i,$$

as required. □

Lemma 4. *$N_{ik} \cap \langle g_i \rangle = \langle g_i^{m_k} \rangle$, $0 \leqq k \leqq i$.*

Proof. First, let us show that the element $b_i \in B_i$ (see (9)) is of order m_i. Indeed, $b_0 = 1$ and we may assume by induction that $b_{i-1} \in B_{i-1}$ is of order m_{i-1}. Then, by an easy calculation,

$$\begin{aligned}b_i^{m_{i-1}} &= (a_i(1)b_{i-1})^{m_{i-1}} \\ &= a_i(1)(b_{i-1}a_i(1)b_{i-1}^{-1})(b_{i-1}^2a_i(1)b_{i-1}^{-2})\cdots(b_{i-1}^{m_{i-1}-1}a_i(1)b_{i-1}^{-(m_{i-1}-1)})b_{i-1}^{m_{i-1}} \\ &= a_i(1)a_i(b_{i-1}^{-1})a_i(b_{i-1}^{-2})\cdots a_i(b_{i-1}^{-(m_{i-1}-1)}).\end{aligned}$$

Each $a_i(b)$ is of order m_i/m_{i-1} and for any $b', b'' \in B_{i-1}$ the elements $a_i(b')$, $a_i(b'')$ commute. Therefore $b_i^{m_{i-1}}$ is of order m_i/m_{i-1} and, consequently, b_i is of order m_i. This completes the induction.

Now consider the homomorphism $\psi_{ik}: G_i \to B_k$. We have Ker $\psi_{ik} = N_{ik}$ and $\psi_{ik}(g_i) = b_k$. It follows that $N_{ik} \cap \langle g_i \rangle = \langle g_i^{m_k} \rangle$. □

Now we can finish the proof of Proposition 2. In view of Lemmas 3 and 4,

$$\{m_0 = 1, m_1, \dots, m_i\} \subseteq I(G_i, g_i) \subseteq \{m_0 = 1, m_1, \dots, m_{i-1}, m_i, 2m_i, 3m_i, \dots\}.$$

On the other hand, $D_i^k \leqq_f G_i$ so that $D_i^k \cap \langle g_i \rangle = \langle g^{km_i} \rangle$ whence condition (2). Condition (3) also follows from Lemmas 3 and 4. We have $G_i/N_{ik} \cong B_k \cong G_j/N_{jk}$ for any $j > i$, so condition (4) holds.

The proposition is proved.

A subsequent publication is planned which will contain an example of a *finitely generated* group with the properties of the title.

Acknowledgement

I thank R. Gitik for valuable information and comments.

References

[AG] R. B. F. T. Allenby and R. F. Gregorac, *On locally extended residually finite groups*, Conference on Group Theory, Lecture Notes in Mathematics **319**, Springer-Verlag, Berlin–Heidelberg–New York, 1973, pp. 9–17.

[Gr] K. W. Gruenberg, *Cohomological Topics in Group Theory*, Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[Ne] H. Neumann, *Varieties of Groups*, Springer-Verlag, Berlin–Heidelberg–New York, 1967.